

NICHOLS-WORONOWICZ ALGEBRA MODEL FOR SCHUBERT CALCULUS ON COXETER GROUPS

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ABSTRACT. We realise the cohomology ring of a flag manifold, more generally the coinvariant algebra of an arbitrary finite Coxeter group W , as a commutative subalgebra of a certain Nichols-Woronowicz algebra in the Yetter-Drinfeld category over W . This gives a braided Hopf algebra version of the corresponding Schubert calculus. The nilCoxeter algebra and its action on the coinvariant algebra by divided difference operators are also realised in the Nichols-Woronowicz algebra. We discuss the relationship between Fomin-Kirillov quadratic algebras, Kirillov-Maeno bracket algebras and our construction.

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0. INTRODUCTION

Few years ago, a new approach to the cohomology rings of the flag manifolds was developed by Fomin and Kirillov. In [FK] they introduced a family of noncommutative graded algebras \mathcal{E}_n , defined only by quadratic relations, such that the cohomology ring of the manifold Fl_n of complete flags in \mathbb{C}^n is realised as a graded commutative subalgebra of \mathcal{E}_n . The Fomin-Kirillov algebras have many other interesting properties, notably they are braided Hopf algebras over the symmetric groups \mathbb{S}_n .

The manifolds Fl_n naturally correspond to the symmetric groups \mathbb{S}_n and have a version where \mathbb{S}_n is replaced by any crystallographic Coxeter group W : if G is a semisimple Lie group whose Weyl group is W , and B is the Borel subgroup of G , the homogeneous space G/B is the flag manifold of G . The cohomology ring of G/B was shown by Borel to be isomorphic to the coinvariant algebra S_W of W . Hence the cohomology of the flag manifold

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has a description purely in terms of the invariant theory of W — we refer to this description as a Schubert calculus over W . If one looks only at the algebraic side of the picture, W does not need be crystallographic (see the book [Hi] of Hiller). Thus, any finite Coxeter group W admits a Schubert calculus, although not necessarily coming from the cohomology of a geometric object.

It is then natural to try and extend the Fomin-Kirillov construction to arbitrary Coxeter groups. Recently, Kirillov and Maeno suggested a generalisation of \mathcal{E}_n , where the symmetric group \mathbb{S}_n is replaced by a finite Coxeter group W with a set S of Coxeter generators. The bracket algebras $BE(W, S)$, defined in [KM1], are in general not quadratic, but the relations in $BE(W, S)$ are still given explicitly in terms of the root system of W . However, the conjecture that $BE(W, S)$ contains a copy of S_W was verified in [KM1] only for classical crystallographic groups W and for W of type G_2 .

In the present paper, we suggest a new and uniform construction for the coinvariant algebra S_W of an arbitrary Coxeter group W . We realise S_W as a graded commutative subalgebra in a Nichols-Woronowicz algebra \mathcal{B}_W , which itself is a braided Hopf algebra over the group W with a number of additional properties. Moreover, the so-called nilCoxeter algebra N_W and its important representation on S_W by divided difference operators, are also found within the same algebra \mathcal{B}_W .

Our point of view of the Nichols-Woronowicz algebras is via the braided group theory developed by Majid in mid-1990s (see e.g. [M1, M2, M3]). The principal idea of Majid's approach is that to each object in a braided category, there is canonically associated a pair of Hopf algebras in this braided category, which are non-degenerately dually paired. We call each of these dually paired braided Hopf algebras a Nichols-Woronowicz (sometimes simply Nichols) algebra.

The term ‘Nichols algebra’ was introduced by Andruskiewitsch and Schneider in [AS1] and refers to an equivalent definition of this object as a graded braided Hopf algebra, generated by its degree one component which is the set of primitives. These conditions first appeared in the work [N] of Nichols. Apparently the first explicit construction of a Nichols-Woronowicz algebra per se appeared in the paper [W] by Woronowicz, where exterior algebras for quantum differential calculi were studied; the term ‘Woronowicz exterior algebra’ is used by a number of authors. The Nichols-Woronowicz algebras seem to become an increasingly popular object of study.

The relations in the Nichols algebra are Woronowicz relations which ensure the non-degeneracy of the duality pairing mentioned above. They are not as explicit as the relations in \mathcal{E}_n or $BE(W, S)$, and in practice, we do not work with the relations directly. We use the methods of braided differential calculus, which makes our approach essentially different from that of [FK] and [KM1].

How to apply these methods to the Fomin-Kirillov algebras in the case of symmetric group, was explicitly shown by Majid in [M5]. In particular, the divided difference operators are interpreted as restrictions of braided partial derivatives — the version of this for an arbitrary Coxeter group plays a central role in Section 5 below. It was also proposed in [M5] to replace the algebras \mathcal{E}_n by their Woronowicz quotient and to extend the construction to other Coxeter groups, which is achieved in the present paper.

Our construction was inspired, besides the papers mentioned above, by the work [MiS] of Milinski and Schneider. In [MiS], a general scheme for constructing Nichols algebras over a Coxeter group is discussed (our algebra \mathcal{B}_W fits into this scheme), and the Nichols algebra $\mathcal{B}_{\mathbb{S}_n}$ is explicitly introduced. Let us also mention a more recent preprint [KM2] where the ‘super’ Nichols-Woronowicz algebras $\Lambda_w(W)$, which control the noncommutative geometry of Weyl groups W , are considered.

The structure of the paper is as follows. In Section 1 we recall basic facts about Coxeter groups, their root systems, coinvariant algebras, nilCoxeter algebras and Schubert classes. The Nichols algebras are defined in Section 3, after a brief exposition of braided differential calculi in Section 2. Our principal example of the Nichols algebra is \mathcal{B}_W , a graded braided Hopf algebra in the Yetter-Drinfeld category over the Coxeter group W , described in Section 4.

After all this preparatory work, in Section 5 we state a principal result of the paper, Theorem 5.4, which implies that \mathcal{B}_W contains a graded subalgebra isomorphic to the coinvariant algebra S_W . This commutative subalgebra is generated by such a subspace U of the degree 1 in \mathcal{B}_W that (i) U is isomorphic, as a W -module, to the reflection representation of W ; (ii) U is ‘generic’. We describe all such U , which we call generic reflection submodules.

It turns out that if W is a crystallographic Coxeter group with a simply laced Dynkin diagram, there is exactly one canonical reflection submodule, which generates a canonical copy of S_W in \mathcal{B}_W . If $W = \mathbb{S}_n$, this reflection submodule is precisely the space generated by Dunkl elements defined in [FK]. Note that we do not use explicit expressions for specific Dunkl elements; that they generate a copy of S_W follows solely from the fact that their span is a reflection representation of W . For a Coxeter group of a non-simply laced type, there always is more than one way to embed S_W into \mathcal{B}_W , but any embedding may be obtained from a given one by composing with an explicit automorphism of \mathcal{B}_W .

Section 6 is devoted to the realisation of the nilCoxeter algebra N_W in \mathcal{B}_W . Thus, two different objects, the coinvariant algebra S_W and the nilCoxeter algebra N_W , are identified with subalgebras in the Nichols-Woronowicz algebra \mathcal{B}_W ; the (right) action of N_W on S_W is interpreted as the natural action of the braided Hopf algebra \mathcal{B}_W on itself by derivations; the non-degenerate pairing between S_W and N_W is just the self-duality pairing on \mathcal{B}_W restricted to these two subalgebras.

The most important observation in Section 6 is that the simple root generators in the Nichols-Woronowicz algebra \mathcal{B}_W obey the Coxeter relations. This fact, interesting by itself, is proved by expressing the Woronowicz symmetriser of both sides of a Coxeter relation in terms of paths in the Bruhat graph of a Coxeter group.

In the last section of the paper we mention that \mathcal{B}_W is a quotient of the Fomin-Kirillov algebra \mathcal{E}_n when $W = \mathbb{S}_n$, and show that \mathcal{B}_W is a quotient of the Kirillov-Maeno bracket algebra for many other Coxeter groups W . We finish by repeating a conjecture made by several authors, that $\mathcal{B}_{\mathbb{S}_n} = \mathcal{E}_n$.

It should be noted that there are q -deformed versions of Fomin-Kirillov and Kirillov-Maeno algebras (see [FK] and [KM1]), which are intended to be a model for small quantum cohomology rings of flag manifolds. A more recent preprint [KM3] by Kirillov and Maeno, which uses the results of the preliminary version of the present paper, suggests an embedding of a quantum cohomology ring for a crystallographic Coxeter group W into a tensor square of the Nichols-Woronowicz algebra \mathcal{B}_W . We have, however, left the issue of quantising the

Nichols-Woronowicz algebra model, as well as questions related to the structural theory of Nichols algebras such as finite dimensionality and Hilbert series, beyond the scope the present paper.

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1. THE COINVARIANT ALGEBRA OF W

1.1. The root system and the reflection representation. Let W denote a finite Coxeter group generated by s_1, \dots, s_r subject to the relations $(s_i s_j)^{m_{ij}} = 1$ ($1 \leq i, j \leq r$), where the integers m_{ij} satisfy $m_{ii} = 1$, $m_{ij} = m_{ji} \geq 2$ for $i \neq j$. The length of an element w of W , denoted by $\ell(w)$, is defined as the smallest possible number l of factors in a decomposition $w = s_{i_1} s_{i_2} \dots s_{i_l}$ of w into a product of Coxeter generators s_i ; any such decomposition with $l = \ell(w)$ is called reduced.

Let \mathfrak{h} be a vector space with a fixed basis $\alpha_1, \dots, \alpha_r$ and a non-degenerate symmetric bilinear form (\cdot, \cdot) defined by $(\alpha_i, \alpha_j) = -\cos(\pi/m_{ij})$. To each $\alpha \in \mathfrak{h}$ satisfying $(\alpha, \alpha) = 1$ there is associated an orthogonal reflection $h \mapsto h - 2(h, \alpha)\alpha$ of \mathfrak{h} . Let the generators s_i of W act on \mathfrak{h} by the reflections associated to α_i ; this gives rise to the reflection representation $W \rightarrow \mathrm{GL}(\mathfrak{h})$, which is faithful. The action of W preserves the bilinear form (\cdot, \cdot) on \mathfrak{h} .

The vectors $\alpha_1, \dots, \alpha_r$ are simple roots; all W -images of the α_i in \mathfrak{h} are roots and form the root system R of W . The construction implies that $(\alpha, \alpha) = 1$ for all $\alpha \in R$. A root which can be written as $\sum_{i=1}^r c_i \alpha_i$, with nonnegative real c_i , is called positive. One has $R = R^+ \sqcup -R^+$ where R^+ is the set of positive roots.

If α is a root, that is, $\alpha = w(\alpha_i)$ for some $w \in W$, $1 \leq i \leq r$, then $s_\alpha := w s_i w^{-1}$ acts on \mathfrak{h} as the reflection associated to α . Therefore, $s_\alpha = s_{-\alpha}$ is a well-defined element of W (which does not depend on the choice of w and i).

This construction of the root system and the reflection (also called geometric) representation of W is given in Part II of [Hu]. The space \mathfrak{h} can be defined over the field of real numbers; we nevertheless consider its complexification and assume the ground field to be \mathbb{C} . The reflection representation of W is irreducible, if and only if W is irreducible as a Coxeter group [B, V.§4.7]. If W is a crystallographic Coxeter group [B, VI.§2.5], the reflection representation \mathfrak{h} of W may be identified with a Cartan subalgebra of a complex semisimple Lie algebra \mathfrak{g} , and W with the Weyl group of \mathfrak{g} .

The W -action on \mathfrak{h} extends to the symmetric algebra $S(\mathfrak{h}) = \bigoplus_{n \geq 0} S^n(\mathfrak{h})$ of \mathfrak{h} . We will refer to the elements of $S(\mathfrak{h})$ as polynomials.

1.2. The coinvariant algebra of W . By a fundamental result of Chevalley [C], the subalgebra $S(\mathfrak{h})^W$ of W -invariant polynomials in $S(\mathfrak{h})$ is itself a free commutative algebra of

rank r . Its generators f_1, \dots, f_r may be chosen to be homogeneous polynomials of degrees $\deg f_i = m_i + 1$, where $1 \leq m_1 \leq \dots \leq m_r$ are integers which depend on W but not on the choice of a particular set of generators. These m_i are called the exponents of the Coxeter group W .

Let I_W be the ideal in $S(\mathfrak{h})$ generated by f_1, \dots, f_r ; in other words, $I_W = S(\mathfrak{h})S(\mathfrak{h})_+^W$ where $S(\mathfrak{h})_+^W$ is the set of W -invariant polynomials without constant term. The coinvariant algebra of W is, by definition,

$$S_W = S(\mathfrak{h})/I_W.$$

The ideal I_W is W -stable and graded, hence S_W is a graded W -module. As shown in [C], there is an ungraded module isomorphism between S_W and the regular representation of W . In particular, the dimension of S_W is equal to the number of elements in W . See [BL] for information on the graded module structure of S_W .

1.3. The nilCoxeter algebra of W . The nilCoxeter algebra N_W of W arises naturally in connection with the coinvariant algebra S_W . Namely, N_W is the algebra generated by the divided difference operators acting on polynomials $f \in S(\mathfrak{h})$ as well as on their classes in S_W . We now recall the abstract definition of N_W and its representation by divided difference operators.

Let N_W be the algebra generated by r generators u_1, \dots, u_r subject to the nilCoxeter relations

$$u_i u_j u_i \dots = u_j u_i u_j \dots \quad (m_{ij} \text{ factors on each side}); \quad u_i^2 = 0.$$

For $w \in W$ with a reduced decomposition $w = s_{i_1} \dots s_{i_l}$, define u_w as the product $u_{i_1} \dots u_{i_l}$. (The element u_w is well-defined because of the Coxeter relations between the u_i .) The elements u_w , $w \in W$, are known to form a homogeneous linear basis of the algebra N_W ; the multiplication table of N_W in terms of this basis is

$$u_v u_w = \begin{cases} u_{vw}, & \text{if } \ell(v) + \ell(w) = \ell(vw); \\ 0, & \text{if } \ell(v) + \ell(w) > \ell(vw). \end{cases}$$

Thus, N_W is a graded algebra of dimension $|W|$.

1.4. Divided difference operators. Let α be a root, and $s_\alpha \in W$ be the corresponding reflection. The linear operator

$$\partial_\alpha: S(\mathfrak{h}) \rightarrow S(\mathfrak{h}), \quad \partial_\alpha f = \frac{f - s_\alpha(f)}{\alpha},$$

is called the divided difference operator. (These operators were introduced independently by Bernstein-Gelfand-Gelfand and by Demazure). One may note that the polynomial $f - s_\alpha(f)$ is always divisible by α , so that the rational function $\partial_\alpha f$ is a polynomial. Since the ideal $I_W \subset S(\mathfrak{h})$ is preserved by the action of ∂_α , the operators ∂_α act on the coinvariant algebra S_W as well.

Write ∂_i for the divided difference operator ∂_{α_i} corresponding to a simple root α_i ($1 \leq i \leq r$). By [Hi, IV.§1–§2], the operators ∂_i satisfy the nilCoxeter relations 1.3; this gives rise to a representation of the nilCoxeter algebra N_W on $S(\mathfrak{h})$ and on S_W , with the generator u_i acting as ∂_i . Either of these representations is faithful.

1.5. The right action of N_W on the coinvariant algebra. We have described, following Hiller [Hi], a left action of the nilCoxeter algebra N_W on S_W . We will now convert it to a right action, using the algebra isomorphism between N_W and its opposite algebra N_W^{op} given on the linear basis by $u_w \mapsto u_{w^{-1}}$. A right action is more convenient for the purpose of Nichols-Woronowicz algebra realisation of N_W and S_W .

Define the right-hand divided difference operators $\overleftarrow{\partial}_\alpha$ by

$$f \overleftarrow{\partial}_\alpha = \partial_\alpha f, \quad f \overleftarrow{\partial}_i = \partial_i f, \quad f \in S_W,$$

and let N_W act on S_W from the right via

$$f u_i = f \overleftarrow{\partial}_i.$$

In terms of the basis $\{u_w\}$ of N_W , this right action is given by $f u_w = \partial_{w^{-1}}(f)$.

Observe that the operator $\overleftarrow{\partial}_\alpha$ obeys the following version of twisted Leibniz rule:

$$(fg) \overleftarrow{\partial}_\alpha = f \cdot (g \overleftarrow{\partial}_\alpha) + (f \overleftarrow{\partial}_\alpha) s_\alpha(g)$$

for $f, g \in S_W$, cf. [Hi, IV.§1]. That is, $\overleftarrow{\partial}_\alpha$ is a $(1, s_\alpha)$ -twisted derivation of the algebra S_W .

1.6. The non-degenerate pairing between S_W and N_W . Let $\epsilon: S_W \rightarrow \mathbb{C}$ be the projection to the degree zero component $S_W^0 = \mathbb{C}$ of S_W (so, ϵf is the ‘constant term’ of f). Consider a bilinear pairing

$$\langle \cdot, \cdot \rangle_{S_W, N_W}: S_W \otimes N_W \rightarrow \mathbb{C}, \quad \langle f, u_w \rangle = \epsilon(f \overleftarrow{\partial}_w).$$

Similarly to [Hi, IV.§1], this pairing is non-degenerate. This, in particular, means that for any non-zero $f \in S_W$ there exists a product $\overleftarrow{\partial} = \overleftarrow{\partial}_{i_1} \dots \overleftarrow{\partial}_{i_l}$ ($l \geq 0$) of right-hand divided difference operators, such that $0 \neq f \overleftarrow{\partial} \in \mathbb{C}$ (this will be used later in 5.13).

1.7. Schubert classes and Schubert polynomials. In the crystallographic case, the Schubert classes in the cohomology of the flag variety correspond to a distinguished linear basis of S_W . This basis $\{\bar{X}_w \mid w \in W\}$, described by Bernstein, Gelfand and Gelfand, and independently by Demazure, may be defined purely algebraically as follows: $\bar{X}_{w_0} = \frac{1}{|W|} \prod_{\gamma \in R^+} \gamma$ where w_0 is the longest element in W , and for an arbitrary $w \in W$ one has $\bar{X}_w = \partial_{w^{-1}w_0} \bar{X}_{w_0}$. In fact, $\{\bar{X}_w \mid w \in W\}$ is the basis of S_W which is dual to $\{u_w \mid w \in W\} \subset N_W$ with respect to the above duality pairing $\langle \cdot, \cdot \rangle_{S_W, N_W}$.

The term ‘Schubert polynomials’ usually refers to a family of elements $X_w \in S(\mathfrak{h})$ which project to $\bar{X}_w \in S_W$ and satisfy as many combinatorial properties of \bar{X}_w as possible. For the symmetric group $W = \mathbb{S}_n$, the Schubert polynomials were introduced by Lascoux and Schützenberger, see the survey in [Mac]; a construction of Schubert polynomials for other classical types was later suggested independently by Billey and Haiman in [BH] and by Fomin and Kirillov in [FK1]. Schubert polynomials are intended to be a realisation of S_W in a free commutative algebra, whereas the approach of [FK], [KM1] and the present paper is to view S_W as a subalgebra in a noncommutative algebra.

2. FREE BRAIDED DIFFERENTIAL CALCULUS

In this section we recall the free braided differential calculus, as introduced by Majid e.g. in [M2]. The proofs can be found in Chapters 9 and 10 of [M3].

2.1. Braided Hopf algebras. Recall that a braiding in a tensor category (\mathcal{C}, \otimes) is a functorial family of isomorphisms $\Psi_{A,B}: A \otimes B \rightarrow B \otimes A$ (where A, B are any two objects in \mathcal{C}), satisfying the ‘hexagon axioms’ $(\text{id}_B \otimes \Psi_{A,C})(\Psi_{A,B} \otimes \text{id}_C) = \Psi_{A,B \otimes C}$ and $(\Psi_{A,C} \otimes \text{id}_B)(\text{id}_A \otimes \Psi_{B,C}) = \Psi_{A \otimes B, C}$. The associativity isomorphisms such as $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, which allow us not to pay attention to the order of brackets in multiple tensor products, are suppressed but are part of the tensor category setup. A braided category, as defined in [JS], is a tensor category equipped with a braiding.

Let us recall the definition of a braided Hopf algebra, as given e.g. in [M1] (see also a self-contained exposition in [M3]). Suppose A, B are algebras in a braided tensor category $(\mathcal{C}, \otimes, \Psi)$, meaning that their product morphisms $\cdot_A \in \text{Hom}(A \otimes A, A)$, $\cdot_B \in \text{Hom}(B \otimes B, B)$ are fixed, as well as the unit morphisms $\eta_A \in \text{Hom}(\mathbb{I}, A)$, $\eta_B \in \text{Hom}(\mathbb{I}, B)$, where \mathbb{I} is the unit object in \mathcal{C} .

The tensor product of A and B in \mathcal{C} is also equipped with an algebra structure in the following way:

$$\cdot_{A \otimes B} = (\cdot_A \otimes \cdot_B) \circ (\text{id}_A \otimes \Psi_{B,A} \otimes \text{id}_B): A \otimes B \otimes A \otimes B \rightarrow A \otimes B; \quad \eta_{A \otimes B} = \eta_A \otimes \eta_B.$$

The resulting braided tensor product algebra is denoted by $A \underline{\otimes} B$.

A braided bialgebra is an object B in $(\mathcal{C}, \otimes, \Psi)$ which is an algebra and a coalgebra (with coproduct Δ and counit ϵ) such that $\Delta: B \rightarrow B \underline{\otimes} B$ is a morphism of algebras. A braided Hopf algebra is a braided bialgebra with antipode $S: B \rightarrow B$. Note that the antipode is braided-antimultiplicative, i.e. $S \circ \cdot = \cdot \circ \Psi_{B,B} \circ (S \otimes S)$.

There is also a standard notion of graded braided Hopf algebra, meaning that $B = \bigoplus_{n \geq 0} B^n$ in \mathcal{C} , and the structure morphisms $\cdot, \eta, \Delta, \epsilon, S$ of B respect this grading.

2.2. Free braided groups. A braided linear space (V, Ψ) is a pair consisting of a linear space V and a linear operator $\Psi: V \otimes V \rightarrow V \otimes V$ obeying the braid equation

$$\Psi_{12} \Psi_{23} \Psi_{12} = \Psi_{23} \Psi_{12} \Psi_{23} \in \text{End } V^{\otimes 3}.$$

We use the standard ‘leg notation’ Ψ_{12} etc. for the action of matrices on tensor powers. The braiding Ψ is assumed to come from a braided category, where the braid equation is a consequence of the hexagon axioms.

Suppose V to be finite-dimensional and Ψ to be invertible. Consider the full tensor algebra $T(V)$ with braiding canonically extended from V by the hexagon axioms. The coproduct $\Delta: T(V) \rightarrow T(V) \underline{\otimes} T(V)$, counit $\epsilon: T(V) \rightarrow \mathbb{C}$ and antipode $S: T(V) \rightarrow T(V)$ are defined by their values on generators:

$$\Delta v = v \otimes 1 + 1 \otimes v, \quad \epsilon v = 0, \quad S v = -v \quad \text{for } v \in V,$$

which makes $T(V)$ a graded braided Hopf algebra, called a ‘free braided group’.

To the linear dual V^* of V with the braiding $\Psi^* \in \text{End}(V^{*\otimes 2}) = \text{End}((V^{\otimes 2})^*)$ there corresponds the graded braided Hopf algebra $T(V^*)$. We denote the coalgebra maps of $T(V^*)$ by the same letters Δ, ϵ, S and use the Sweedler notation $\Delta a = a_{(1)} \otimes a_{(2)}$.

2.3. Braided duality pairing. The evaluation pairing $\langle \xi, v \rangle = \xi(v)$ between V^* and V may be extended to a pairing

$$\langle \cdot, \cdot \rangle: T(V^*) \otimes T(V) \rightarrow \mathbb{C}$$

satisfying the axioms of duality pairing of (braided) Hopf algebras [M4, (5)]:

$$\begin{aligned}\langle \phi\psi, x \rangle &= \langle \phi, x_{(2)} \rangle \langle \psi, x_{(1)} \rangle, & \langle \phi, xy \rangle &= \langle \phi_{(2)}, x \rangle \langle \phi_{(1)}, y \rangle, \\ \langle 1, x \rangle &= \epsilon x, & \langle \phi, 1 \rangle &= \epsilon \phi, & \langle S\phi, x \rangle &= \langle \phi, Sx \rangle.\end{aligned}$$

This braided duality pairing depends on Ψ and does not coincide with the standard pairing between tensor powers. However, one necessarily has $\langle V^{*\otimes m}, V^{\otimes n} \rangle = 0$ unless $m = n$.

2.4. Braided partial derivatives. The space V^* acts on $T(V)$ via left braided partial derivatives [M4], [M3, 10.4]:

$$\xi \in V^* \mapsto D_\xi: T(V) \rightarrow T(V), \quad D_\xi f = \langle \xi, f_{(1)} \rangle f_{(2)}.$$

Similarly, one has a right action of V on $T(V^*)$ defined by

$$v \in V \mapsto \overleftarrow{D}_v: T(V^*) \rightarrow T(V^*), \quad \phi \overleftarrow{D}_v = \phi_{(1)} \langle \phi_{(2)}, v \rangle.$$

In the present paper, we use right derivatives \overleftarrow{D}_v (Remark 5.10 below explains this choice and discusses another possible choice of derivatives). Here are some of the properties of the \overleftarrow{D}_v . The operator \overleftarrow{D}_v on $T(V^*)$ is adjoint to the multiplication by v from the left in $T(V)$, i.e. $\langle \phi \overleftarrow{D}_v, x \rangle = \langle \phi, vx \rangle$. For $x = v_1 \otimes v_2 \otimes \cdots \otimes v_m \in V^{\otimes m}$, denote $\phi \overleftarrow{D}_x = \phi \overleftarrow{D}_{v_1} \cdots \overleftarrow{D}_{v_m}$. One can then obviously rewrite the braided duality pairing as

$$\langle \phi, x \rangle = \epsilon(\phi \overleftarrow{D}_x).$$

2.5. Braided Leibniz rule. One may use an equivalent definition of operators \overleftarrow{D}_v via the condition $\xi \overleftarrow{D}_v = \xi(v) \in T^0(V^*) = \mathbb{C}$ for $\xi \in T^1(V^*) = V^*$ and the braided Leibniz rule

$$(\phi\psi) \overleftarrow{D}_v = \phi(\psi \overleftarrow{D}_v) + \phi \Psi_{V, T(V^*)}^{-1}(\psi \otimes \overleftarrow{D}_v), \quad \phi, \psi \in T(V^*).$$

Let us explain the notation used in this formula. The operator $\Psi_{V, T(V^*)}$ is the braiding between V and $T(V^*)$ in the braided category \mathcal{C} . It can be expressed solely in terms of the braiding $\Psi_{V, V}$ on V [M3, Proposition 10.3.6], however, we will not use this explicit expression in the general case. (In 4.6, we will use the braided Leibniz rule for a particular braided space V with a simple formula for $\Psi_{V, T(V^*)}$.) Now $\phi \Psi_{V, T(V^*)}^{-1}(\psi \otimes \overleftarrow{D}_v)$ is interpreted in the following way: one applies the inverse braiding $\Psi_{V, T(V^*)}^{-1}$ to $\psi \otimes v$ and obtains an element, say $\sum_i v_i \otimes \psi_i$, of $V \otimes T(V^*)$; one then computes $\sum_i (\phi \overleftarrow{D}_{v_i}) \psi_i \in T(V^*)$.

2.6. Braided symmetriser. The above duality pairing $\langle \cdot, \cdot \rangle$ can be written down explicitly as follows.

Let \mathbb{B}_n denote the braid group with braid generators $\sigma_1, \dots, \sigma_{n-1}$, and let \mathbb{S}_n be the corresponding symmetric group generated by Coxeter generators s_1, \dots, s_{n-1} . The Matsumoto section $t: \mathbb{S}_n \rightarrow \mathbb{B}_n$ is a set-theoretical map defined by the rule $t(\pi) = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_l}$, whenever $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced decomposition of $\pi \in \mathbb{S}_n$. The element $\Sigma_n = \sum_{\pi \in \mathbb{S}_n} t(\pi) \in \mathbb{C}\mathbb{B}_n$ is called the braided (or quantum) symmetriser.

For $k = 1, 2, \dots$, the braided integers $[k]_\sigma \in \mathbb{CB}_n$, the ‘shifted’ braided integers $[k]_\sigma^{(s)} \in \mathbb{CB}_n$, and the braided factorials $[k]!_\sigma \in \mathbb{CB}_n$ are:

$$\begin{aligned} [k]_\sigma &= 1 + \sigma_1 + \sigma_2\sigma_1 + \dots + \sigma_{k-1} \dots \sigma_2\sigma_1, \\ [k]_\sigma^{(s)} &= 1 + \sigma_s + \sigma_{s+1}\sigma_s + \dots + \sigma_{s+k-2} \dots \sigma_{s+1}\sigma_s, \\ [k]!_\sigma &= [k]_\sigma [k-1]_\sigma^{(2)} [k-2]_\sigma^{(3)} \dots [2]_\sigma^{(k-1)}, \end{aligned}$$

cf. [M2]. The braided symmetriser factorises as $\Sigma_n = [n]!_\sigma$.

Let the generator σ_i of \mathbb{B}_n act on $V^{\otimes n}$ as $\Psi_{i,i+1}$. Denote the resulting action of braided integers, resp. braided factorials, by $[k]_\Psi$, resp. $[k]!_\Psi \in \text{End } V^{\otimes n}$. Then the duality pairing 2.3 between $V^{*\otimes n}$ and $V^{\otimes n}$ is explicitly given by

$$\langle \phi, x \rangle = (\phi \mid [n]!_\Psi x) = ([n]!_{\Psi^*} \phi \mid x),$$

where $(\cdot \mid \cdot)$ is the evaluation pairing $(\xi_n \otimes \dots \otimes \xi_2 \otimes \xi_1 \mid v_1 \otimes v_2 \otimes \dots \otimes v_n) = \prod_{i=1}^n \xi_i(v_i)$.

3. NICHOLS-WORONOWICZ ALGEBRAS

3.1. Definition of the Nichols-Woronowicz algebra. Let V be a linear space with a braiding Ψ , and let $T(V)$, $T(V^*)$ be the braided Hopf algebras introduced in the previous section. The duality pairing $\langle \cdot, \cdot \rangle: T(V^*) \times T(V) \rightarrow \mathbb{C}$ may be degenerate. Let $I(V^*)$, resp. $I(V)$, be the kernel of the pairing in $T(V^*)$, resp. $T(V)$. Since we deal with a duality pairing between graded (braided) Hopf algebras, the kernels $I(V^*)$, $I(V)$ are graded Hopf ideals.

The algebras

$$\mathcal{B}(V^*) = T(V^*)/I(V^*), \quad \mathcal{B}(V) = T(V)/I(V)$$

are called the Nichols-Woronowicz algebras of V^* and V .

We now state the basic properties of the Nichols-Woronowicz (also called Nichols) algebra which follow directly from the construction outlined above. For a braided space V , this construction leads to a dual pair of Nichols algebras $\mathcal{B}(V^*)$, $\mathcal{B}(V)$; we formulate the properties for $\mathcal{B}(V^*)$, keeping in mind that their analogues hold for $\mathcal{B}(V)$ as well.

3.2. Lemma. (i) $\mathcal{B}(V^*) = \bigoplus_{n \geq 0} \mathcal{B}(V^*)^n$ is a graded braided Hopf algebra.

(ii) $\mathcal{B}(V^*)^0 = \mathbb{C}$, $\mathcal{B}(V^*)^1 = V^*$.

(iii) There is a Hopf algebra duality pairing $\langle \cdot, \cdot \rangle: \mathcal{B}(V^*) \otimes \mathcal{B}(V) \rightarrow \mathbb{C}$, which is non-degenerate.

(iv) $\mathcal{B}(V^*)$ is generated by $\mathcal{B}(V^*)^1$ as an algebra.

Proof. The ideal $I(V^*)$ is graded, hence (i); the duality pairing $\langle \cdot, \cdot \rangle$ between $T(V^*)$ and $T(V)$ is non-degenerate in $T^0(V^*) = \mathbb{C}$ and $T^1(V^*) = V^*$, hence (ii). The meaning of the construction of $\mathcal{B}(V^*)$ is that one eliminates the kernel of the duality pairing, so (iii) follows; (iv) is obvious since $\mathcal{B}(V^*)$ is a quotient of $T(V^*)$. \square

3.3. Woronowicz relations. Description 2.6 of the duality pairing $\langle \cdot, \cdot \rangle$ means that, in degree n , the kernel of the pairing is precisely the kernel of the braided symmetriser Σ_n . Thus we arrive at the following presentation of the Nichols algebras:

$$\mathcal{B}(V^*) = \bigoplus_{n \geq 0} V^{*\otimes n} / \ker[n]!_{\Psi^*}, \quad \mathcal{B}(V) = \bigoplus_{n \geq 0} V^{\otimes n} / \ker[n]!_\Psi.$$

This presentation (with $-\Psi$ instead of Ψ which does not affect the braid equation) was used as the definition of the exterior algebra for quantum differential calculi in the work of Woronowicz [W].

The relations in $\mathcal{B}(V^*)$, $\mathcal{B}(V)$, although given as kernels of quantum (anti) symmetrisers, are in general not known explicitly and may be complicated. One may try to find the rank of $[n]!_\Psi$ which is the dimension of $\mathcal{B}(V)^n$, or to check if $\mathcal{B}(V)$ is at all finite-dimensional, but this is usually difficult even when the braided space V is ‘small’. An excellent example of this kind of work in the case $\dim V = 2$ is the paper [He] of Heckenberger.

3.4. An equivalent definition of the Nichols algebra. The earliest occurrence of the following properties, characterising the Nichols algebra, was in the work of Nichols [N]: (1) $\mathcal{B}(V)^0 = \mathbb{C}$, $\mathcal{B}(V)^1 = V$; (2) $\mathcal{B}(V)^1 = P(\mathcal{B}(V))$; (3) $\mathcal{B}(V)^1$ generates $\mathcal{B}(V)$ as an algebra. Here $P(\mathcal{B}(V))$ is the set of primitive elements of $\mathcal{B}(V)$ (i.e. those $a \in \mathcal{B}(V)$ satisfying $\Delta a = a \otimes 1 + 1 \otimes a$). A proof of equivalence between this definition and, say, Woronowicz presentation 3.3 can be found in [S].

3.5. Braided partial derivatives. For $v \in V$, one has the braided partial derivative $\overleftarrow{D}_v \in \text{End } T(V^*)$ which satisfies $\langle \phi \overleftarrow{D}_v, x \rangle = \langle \phi, vx \rangle$. It follows that if $\phi \in T(V^*)$ lies in the kernel of the pairing $\langle \cdot, \cdot \rangle$, then $\phi \overleftarrow{D}_v$ does. Therefore, \overleftarrow{D}_v are well-defined endomorphisms of the Nichols-Woronowicz algebra $\mathcal{B}(V^*)$.

The right action of vectors $v \in V$ on $\mathcal{B}(V^*)$ by braided partial derivatives \overleftarrow{D}_v extends to a right action of the algebra $\mathcal{B}(V)$ on $\mathcal{B}(V^*)$. In fact, the operators \overleftarrow{D}_x defined for arbitrary $x \in \mathcal{B}(V)$ as $\phi \overleftarrow{D}_x = \phi_{(1)} \langle \phi_{(2)}, x \rangle$, satisfy

$$\phi \overleftarrow{D}_{xy} = (\phi \overleftarrow{D}_x) \overleftarrow{D}_y.$$

The duality pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$ is induced from $T(V^*)$, $T(V)$, and is therefore given by $\langle \phi, x \rangle = \epsilon(\phi \overleftarrow{D}_x)$, where $\phi \in \mathcal{B}(V^*)$, $x \in \mathcal{B}(V)$. Non-degeneracy of this pairing obviously implies that the right action of $\mathcal{B}(V)$ on $\mathcal{B}(V^*)$ is faithful.

The following criterion, which describes the joint kernel of all braided partial derivatives in $\mathcal{B}(V^*)$, turns out to be extremely useful when working with Nichols-Woronowicz algebras. It asserts that a ‘function’, all of whose partial derivatives are zero, is a constant. The criterion is equivalent to the non-degeneracy of $\langle \cdot, \cdot \rangle$, thus is automatic in the braided differential calculus given by $\mathcal{B}(V^*)$, $\mathcal{B}(V)$; a form of it in the classical Hopf algebra theory approach can be traced back to Nichols [N].

3.6. Criterion. *The following are equivalent: (a) $\phi \in \mathcal{B}(V^*)$ is a constant. (b) $\phi \overleftarrow{D}_v = 0$ for all $v \in V$.*

Proof. If $\phi \in \mathcal{B}(V^*)^0 = \mathbb{C}$, then obviously $\phi \overleftarrow{D}_v = 0$ since \overleftarrow{D}_v lowers the degree by one. Suppose now that a non-zero $\phi \in \mathcal{B}(V^*)^n$ is in the kernel of all \overleftarrow{D}_v . By the non-degeneracy of the pairing, there exists $x \in \mathcal{B}(V)^n$ such that $\langle \phi, x \rangle \neq 0$; since $\mathcal{B}(V)$ is generated by V as an algebra, one may choose x to be $v_1 v_2 \dots v_n$ for some $v_i \in V$. Then $\epsilon(\phi \overleftarrow{D}_{v_1} \dots \overleftarrow{D}_{v_n}) \neq 0$. But since $\phi \overleftarrow{D}_v = 0$ for any v , this can only be possible if $n = 0$. \square

3.7. Two simplest examples. The two simplest examples of Nichols algebras, for any linear space V , are: (a) $\text{Sym}(V)$, the symmetric algebra of V (let $\Psi = \tau$ to be the flip $\tau(x \otimes y) = y \otimes x$); (b) $\bigwedge V$, the exterior algebra of V (let $\Psi = -\tau$). Both cases are observed easily by 3.3 if one notes that the Woronowicz symmetriser $[n]!_\Psi$ becomes the usual (a) symmetrisation, (b) antisymmetrisation map on $V^{\otimes n}$. In the case of symmetric algebra, both left and right braided derivatives D_v, \overleftarrow{D}_v coincide with the usual directional derivative $\frac{\partial}{\partial v}$ on the polynomial ring; on $\bigwedge^n V$, the left braided derivative D_v is the contraction (or interior product) operator corresponding to v and differs from \overleftarrow{D}_v by a factor of $(-1)^{n-1}$.

4. NICHOLS ALGEBRAS \mathcal{B}_W OVER COXETER GROUPS

In this section, our main example of Nichols-Woronowicz algebra is introduced. The braided category it comes from, is the Yetter-Drinfeld module category over the Coxeter group W , whose definition we now recall.

4.1. The Yetter-Drinfeld module category over a finite group. Let Γ be a finite group. The objects of the Yetter-Drinfeld module category ${}^\Gamma\mathcal{YD}$ over Γ are linear spaces V with the following structure:

- (1) the Γ -action $\Gamma \times V \rightarrow V$, $(g, v) \mapsto gv$;
- (2) the $\mathbb{C}\Gamma$ -coaction, which is the same as Γ -grading $V = \bigoplus_{g \in \Gamma} V_g$;
- (3) the compatibility condition $gV_h = V_{ghg^{-1}}$.

One knows that ${}^\Gamma\mathcal{YD}$ is a braided tensor category: for $U, V \in \text{Ob}({}^\Gamma\mathcal{YD})$, the Γ -action on $U \otimes V$ is $g(u \otimes v) = gu \otimes gv$ and the Γ -grading is $(U \otimes V)_g = \bigoplus_{h \in \Gamma} U_h \otimes V_{h^{-1}g}$. For $V \in \text{Ob}({}^\Gamma\mathcal{YD})$, the braiding is given by $\Psi(x \otimes y) = gy \otimes x$ whenever $x \in V_g, y \in V$.

Of course, the general theory of Nichols algebras applies well for this particular type of braided linear spaces. The Nichols algebras in the Yetter-Drinfeld category over a group have been an object of extensive study, in particular because they are linked with pointed Hopf algebras — see for example the survey [AS2] of Andruskiewitsch and Schneider.

4.2. The Yetter-Drinfeld module V_W . We now specify Γ to be the Coxeter group W and will introduce a particular braided space $V_W \in \text{Ob}({}_W^W\mathcal{YD})$, thus linking the content of Section 1 with the Nichols algebra theory of Sections 2 and 3. We will freely use the notation from all preceding sections.

Let V_W be the linear space spanned by symbols $[\alpha]$ where α is a root of W , subject to the relation $[-\alpha] = -[\alpha]$. The dimension of V_W is thus $|R^+|$.

The W -action on V_W is given by $w[\alpha] = [w\alpha]$, and the W -grading is given by assigning the degree s_α to the basis element $[\alpha]$. The action and the grading are compatible, so that V_W is a Yetter-Drinfeld module over W . The resulting braiding Ψ on V_W is given explicitly by

$$\Psi([\alpha] \otimes [\beta]) = [s_\alpha \beta] \otimes [\alpha].$$

Our main object is the Nichols algebra $\mathcal{B}(V_W)$.

4.3. Remark. The definition of V_W in fact comes from at least two sources.

First, precisely this linear space is the degree 1 component in the bracket algebra $BE(W, S)$, defined by Kirillov and Maeno in [KM1]. The algebra $BE(W, S)$ has the same quadratic

relations as $\mathcal{B}(V_W)$, but in general, bracket algebras of [KM1] have less relations than $\mathcal{B}(V_W)$ and are not Nichols algebras. (See 7.3 below for more details.)

Second, there is a recipe by Milinski and Schneider [MiS, section 5], which, applied to the Coxeter group W , suggests to take the vector space V with basis $\{x_t\}_{t \in T}$ where T is the set of all reflections in W . The basis element x_t is of W -degree t . The W -action on V is given by $gx_t = \chi(g, t)x_{gtg^{-1}}$, where the function $\chi: W \times T \rightarrow \mathbb{C}$ satisfies $\chi(gh, t) = \chi(g, hth^{-1})\chi(h, t)$, so that V is a Yetter-Drinfeld module.

Example 5.3 in [MiS] defines χ in the case $W = \mathbb{S}_n$ by $\chi(g, t) = 1$ if $g(i) < g(j)$, $\chi(g, t) = -1$ if $g(i) > g(j)$, where $t = (ij)$ is a reflection, $1 \leq i < j \leq n$, and $g \in \mathbb{S}_n$. Our Yetter-Drinfeld module V_W generalises this example to the case of an arbitrary Coxeter group: indeed, put $x_{s_\alpha} = [\alpha]$ for $\alpha \in R^+$, and define the function χ by

$$\alpha \in R^+, g \in W \quad \Rightarrow \quad \chi(g, x_{s_\alpha}) = \begin{cases} 1, & \text{if } g(\alpha) \in R^+; \\ -1, & \text{if } -g(\alpha) \in R^+. \end{cases}$$

4.4. The self-dual Nichols algebra \mathcal{B}_W . In the general picture of Section 3, we had two dually paired Nichols algebras $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$. In the case $V = V_W$, however, it is useful to identify V_W with its dual V_W^* via the non-degenerate bilinear form on V_W defined by

$$\langle [\alpha], [\beta] \rangle = \delta_{\alpha, \beta}, \quad \alpha, \beta \in R^+.$$

This bilinear form is W -invariant and is compatible, in a proper sense, with the W -grading on V_W . Thus, V_W and V_W^* are isomorphic as objects in the category ${}^W_W\mathcal{YD}$. The braiding $\Psi \in \text{End } V_W^{\otimes 2}$ becomes self-adjoint with respect to the evaluation pairing on $V_W^{\otimes 2}$. The Nichols algebras $\mathcal{B}(V_W^*)$ and $\mathcal{B}(V_W)$ are then canonically identified:

$$\mathcal{B}_W := \mathcal{B}(V_W^*) = \mathcal{B}(V_W).$$

Note that \mathcal{B}_W , being a braided Hopf algebra in ${}^W_W\mathcal{YD}$, is a W -module. The W -action is given by $w([\alpha_1][\alpha_2] \dots [\alpha_n]) = [w\alpha_1][w\alpha_2] \dots [w\alpha_n]$.

4.5. Braided partial derivatives in \mathcal{B}_W . To each positive root α there corresponds, by the general construction outlined above and the self-duality of \mathcal{B}_W , the right braided partial derivative $\overleftarrow{D}_{[\alpha]}$ on \mathcal{B}_W . Let us restate the main properties of braided partial derivatives for this particular Nichols algebra.

First, the restriction of $\overleftarrow{D}_{[\alpha]}$ onto $V_W = \mathcal{B}_W^1$ is given by

$$[\beta] \overleftarrow{D}_{[\alpha]} = \begin{cases} \pm 1, & \alpha = \pm \beta, \\ 0, & \alpha \neq \pm \beta, \end{cases} \quad \beta \in R.$$

4.6. Second, braided Leibniz rule 2.5 simplifies: since V_W is self-dual as was shown 4.4, the braiding Ψ_{V_W, V_W^*} is now the same as $\Psi_{V_W, V_W}: [\alpha] \otimes f \mapsto s_\alpha(f) \otimes [\alpha]$ for $\alpha \in R, f \in V_W$. The inverse braiding is thus given by $\Psi_{V_W, V_W}^{-1}(f \otimes [\alpha]) = [\alpha] \otimes s_\alpha(f)$. It follows from the hexagon axiom that $\Psi_{V_W, T(V_W)}^{-1}$ is given by the same formula (but with $f \in T(V_W)$); passing to the quotient, one obtains the same inverse braiding between V_W and \mathcal{B}_W . The braided Leibniz rule now becomes

$$(fg) \overleftarrow{D}_{[\alpha]} = f(g \overleftarrow{D}_{[\alpha]}) + (f \overleftarrow{D}_{[\alpha]})s_\alpha(g), \quad f, g \in \mathcal{B}_W, \alpha \in R.$$

That is, $\overleftarrow{D}_{[\alpha]}$ is an $(1, s_\alpha)$ -twisted derivation of \mathcal{B}_W .

Finally, 3.6 now gives the following

4.7. Criterion. *$f \in \mathcal{B}_W$ is a constant, if and only if $f \overleftarrow{D}_{[\alpha]} = 0$ for all $\alpha \in R^+$.*

5. THE REALISATION OF THE COINVARIANT ALGEBRA IN \mathcal{B}_W

In this section we describe a graded subalgebra of \mathcal{B}_W isomorphic to the coinvariant algebra of the Coxeter group W . Thus, the Nichols algebra \mathcal{B}_W provides a model for the coinvariant algebra, Schubert calculus and (in the crystallographic case) cohomology of the flag manifold for an arbitrary Coxeter group W in the same sense as the Fomin-Kirillov algebras \mathcal{E}_n from [FK] provide such a model for $W = \mathbb{S}_n$. The relationship between \mathcal{B}_W and the Fomin-Kirillov algebras will be discussed in the next Section.

The degree-preserving W -equivariant embedding $S_W \hookrightarrow \mathcal{B}_W$ turns out to be unique up to a composition with certain automorphisms of \mathcal{B}_W , which we describe explicitly.

5.1. Reflection submodules in V_W . The algebra S_W is generated by its degree 1 component $S^1(\mathfrak{h}) = \mathfrak{h}$, which is the reflection representation of W . Hence, a graded subalgebra of \mathcal{B}_W which is isomorphic to S_W as a graded algebra, must be generated by $U \subset V_W = \mathcal{B}_W^1$, such that U is a W -submodule of V_W isomorphic to the reflection representation \mathfrak{h} of W .

We will, however, be slightly more general and consider all non-zero submodules $U \subset V_W$ which are images of W -homomorphisms $\mu: \mathfrak{h} \rightarrow V_W$. Such submodules U of V_W will be called reflection submodules.

5.2. The support of a submodule. Before we describe subalgebras generated by reflection submodules, let us introduce a bit more notation. Let U be a linear subspace of V_W . Define the support of U by

$$\text{supp } U = \{\alpha \in R \mid U \overleftarrow{D}_{[\alpha]} \neq 0\}.$$

In other words, the support of U is the minimal set of roots $\pm\alpha$, such that the linear span of $[\alpha]$ in V_W contains U . If U is a W -invariant subspace of V_W , then $\text{supp } U$ is a W -invariant subset of R , and therefore is itself a root system in \mathfrak{h} . Let $W(\text{supp } U) \subseteq W$ be the group generated by reflections with respect to the roots in $\text{supp } U$.

5.3. Generic reflection submodules. We call a submodule $U \subset V_W$ generic, if $\text{supp } U$ is the whole of R . The generic reflection submodules are singled out by the condition $W(\text{supp } U) = W$. We justify the term ‘generic’ later in 5.6.

The rest of this section will mainly be devoted to the proof of the following

5.4. Theorem. (i) *Let U be a reflection submodule in V_W . The subalgebra generated by U in \mathcal{B}_W is commutative. It is isomorphic, as a graded algebra, to the coinvariant algebra of the Coxeter group $W(\text{supp } U)$.*

(ii) *Generic reflection submodules of V_W exist. Each such submodule generates a subalgebra of \mathcal{B}_W isomorphic to S_W .*

Let us start with a Lemma which provides an explicit description of reflection submodules in V_W .

5.5. Lemma. (1) Any W -module homomorphism $\mu \in \text{Hom}_W(\mathfrak{h}, V_W)$ is given by a formula

$$\mu(x) = \sum_{\alpha \in R} c_\alpha(x, \alpha)[\alpha],$$

where $\alpha \mapsto c_\alpha$ is a W -invariant scalar function on the root system R .

(2) The support of $\mu(\mathfrak{h})$ is $\{\alpha \in R \mid c_\alpha \neq 0\}$.

(3) μ is injective if and only if $\text{supp } \mu(\mathfrak{h})$ spans \mathfrak{h} .

Proof. (1) Consider a new W -module \widetilde{V}_W , with linear basis $\{v_\alpha \mid \alpha \in R\}$ and the W -action given by $wv_\alpha = v_{w\alpha}$. The module V_W is a submodule of \widetilde{V}_W , via the inclusion $[\alpha] = v_\alpha - v_{-\alpha}$. For any linear map μ from \mathfrak{h} to \widetilde{V}_W , there are elements b^α of \mathfrak{h} such that $\mu(x) = \sum_{\alpha \in R} (x, b^\alpha)v_\alpha$. The map μ is W -equivariant, if and only if $b^{w\alpha} = wb^\alpha$ for any root α and any element w of the group W .

Now $\text{Hom}_W(\mathfrak{h}, V_W)$ will consist of those W -maps $\mu: \mathfrak{h} \rightarrow \widetilde{V}_W$ whose image lies in V_W . In terms of the elements b^α this translates to $b^{-\alpha} = -b^\alpha$. On the other hand, let $s_\alpha \in W$ be the reflection associated to the root α ; since $s_\alpha\alpha = -\alpha$, one has $s_\alpha b^\alpha = b^{-\alpha} = -b^\alpha$. This immediately implies that b^α is proportional to α , say $b^\alpha = c_\alpha\alpha$, and the W -equivariance of the b^α implies that $c_{w\alpha} = c_\alpha$ for any element w of W ; so (1) follows. (2) is immediate from the definition 5.2 of support. The kernel of μ consists of those $x \in \mathfrak{h}$ which are orthogonal to all $c_\alpha\alpha$, i.e. $\ker \mu = (\text{supp } \mu(\mathfrak{h}))^\perp$, hence (3). \square

5.6. We now have the following information on reflection submodules in V_W , immediate from Lemma 5.5. Let, say, RS be the variety of all reflection submodules of V_W , and let $RS_{\cong \mathfrak{h}}$ (resp. RS_{gen}) be the part of RS consisting of submodules isomorphic to \mathfrak{h} (resp. generic submodules). First of all, RS is not empty and is of dimension equal to the number of W -orbits in R minus one. Furthermore, $RS_{\cong \mathfrak{h}} \supseteq RS_{\text{gen}}$; the generic part RS_{gen} , as well as $RS_{\cong \mathfrak{h}}$, is an open dense set in RS (any W -invariant function $\alpha \mapsto c_\alpha$ on R , such that $c_\alpha \neq 0$ for all α , gives rise to a generic reflection submodule). In particular, generic reflection submodules of V_W exist.

5.7. The multiplicity of \mathfrak{h} in V_W . If W is an irreducible Coxeter group [B, IV.§1.9], the reflection representation \mathfrak{h} is irreducible [B, V.§4.7-8]. The multiplicity of \mathfrak{h} in V_W is then equal to the number of W -orbits in the root system R . If, moreover, W is a Weyl group of simply laced type (so that there is only one orbit in R), then there is a canonical non-zero reflection submodule in V_W , which is generic.

If $W = \mathbb{S}_n$, this canonical reflection submodule is precisely the subspace spanned by Dunkl elements in the terminology of [FK].

Our next step is to establish the commutativity of subalgebras in \mathcal{B}_W generated by reflection submodules.

5.8. Proposition. Let U be a reflection submodule of V_W . The subalgebra $\langle U \rangle$ of $\mathcal{B}(V_W)$, generated by U , is commutative.

Proof. Let U be the image of a W -module map $\mu: \mathfrak{h} \rightarrow V_W$. We will show that any two elements of U commute in \mathcal{B}_W . By the formula for μ given in Lemma 5.5, two elements of U can be written as $\mu(x) = \sum_{\alpha \in R} c_\alpha(x, \alpha^\vee)[\alpha]$ and similarly $\mu(y)$. The commutator

$[\mu(x), \mu(y)]$ is an element of degree 2 in \mathcal{B}_W . According to presentation 3.3 of \mathcal{B}_W , the commutator vanishes if and only if

$$(\text{id} + \Psi)(\mu(x) \otimes \mu(y)) = (\text{id} + \Psi)(\mu(y) \otimes \mu(x)),$$

where Ψ is the braiding 4.2 of V_W . The left hand side rewrites as

$$\sum_{\alpha, \beta \in R} c_\alpha c_\beta ((x, \alpha)(y, \beta) + (x, \beta)(y, s_\beta \alpha)) [\alpha] \otimes [\beta];$$

since $s_\beta \alpha = \alpha - 2(\alpha, \beta)\beta$, this equals

$$\sum_{\alpha, \beta \in R} c_\alpha c_\beta ((x, \alpha)(y, \beta) + (x, \beta)(y, \alpha) - 2(x, \beta)(y, \beta)(\alpha, \beta)) [\alpha] \otimes [\beta].$$

This expression for the left hand side is symmetric in x and y , therefore is equal to the right hand side. \square

It follows from the last Proposition that any W -homomorphism $\mu: \mathfrak{h} \rightarrow V_W$ extends to a map $\mu: S(\mathfrak{h}) \rightarrow \mathcal{B}_W$ of W -module algebras. The kernel of μ will be calculated using the vanishing criterion 4.7, but for that we need to know how to apply the braided derivations $\overleftarrow{D}_{[\alpha]}$ to $\mu(f)$, $f \in S(\mathfrak{h})$. The following key Lemma, which is ideologically the same as Proposition 9.5 from [FK], shows how to express $\mu(f)\overleftarrow{D}_{[\alpha]}$ in terms of the divided difference operator $\overleftarrow{\partial}_\alpha$ acting on $S(\mathfrak{h})$.

5.9. Lemma. *Suppose that an algebra homomorphism $\mu: S(\mathfrak{h}) \rightarrow \mathcal{B}_W$ is defined by $\mu(x) = \sum_{\beta \in R} c_\beta(x, \beta)[\beta]$ for $x \in \mathfrak{h}$. Then*

$$\mu(f)\overleftarrow{D}_{[\alpha]} = c_\alpha \mu(f\overleftarrow{\partial}_\alpha) \quad \text{for } f \in S(\mathfrak{h}), \alpha \in R.$$

Proof. The maps $F_1(f) = \mu(f)\overleftarrow{D}_{[\alpha]}$, $F_2(f) = c_\alpha \mu(f\overleftarrow{\partial}_\alpha)$ from $S(\mathfrak{h})$ to \mathcal{B}_W vanish on constants. Apply them to $x \in \mathfrak{h} = S^1(\mathfrak{h})$. By 4.5, $F_1(x) = c_\alpha(x, \alpha)[\alpha] + c_{-\alpha}(x, -\alpha)[- \alpha] = 2c_\alpha(x, \alpha)$; since $x\overleftarrow{\partial}_\alpha = 2(x, \alpha)$, one has $F_2(x) = 2c_\alpha(x, \alpha)$, hence F_1 and F_2 agree on \mathfrak{h} . By 4.6 and 1.4, both are extended to products of elements of \mathfrak{h} according to the twisted Leibniz rule $F_i(fg) = \mu(f)F_i(g) + F_i(f)\mu(s_\alpha(g))$. Therefore, $F_1 = F_2$. \square

5.10. Remark. It is this lemma that explains why we chose the right partial derivatives $\overleftarrow{D}_{[\alpha]}$ over their seemingly more convenient left-hand counterparts $D_{[\alpha]}$. The advantage of $\overleftarrow{D}_{[\alpha]}$ is that this satisfies the $(1, s_\alpha)$ -twisted Leibniz rule, and as a consequence, coincides, up to a scalar factor, with the divided difference operator ∂_α on S_W . The left derivatives $D_{[\alpha]}$ on \mathcal{B}_W do not obey such a reasonable Leibniz rule.

Another choice of braided partial derivatives to realise the divided difference operators would be the braided-left derivatives \bar{D}_ξ , $\xi \in V^*$ which are defined on $T(V)$, for an arbitrary braided space V , by $\bar{D}_\xi f = (\text{id}_{T(V)} \otimes \langle \cdot, \cdot \rangle)(\Psi_{V^*, T(V)}(\xi \otimes f_{(1)}) \otimes f_{(2)})$. The derivatives $\bar{D}_{[\alpha]}$ were used in the case $W = \mathbb{S}_n$ in [M5]. The operator $\bar{D}_{[\alpha]}$ on \mathcal{B}_W satisfies the $(s_\alpha, 1)$ -twisted Leibniz rule which is equally good for the divided difference operator ∂_α . The only drawback for us is that the $\bar{D}_{[\alpha]}$ do not give rise to a representation of \mathcal{B}_W on itself. They lead to an action of a new algebra $\tilde{\mathcal{B}}_W$, with the same underlying linear space as \mathcal{B}_W but with twisted multiplication $f \star g = \cdot \circ \Psi_{\mathcal{B}_W, \mathcal{B}_W}^{-1}(f \otimes g)$. Using $\bar{D}_{[\alpha]}$ instead of $\overleftarrow{D}_{[\alpha]}$, one can modify

the content of this and the next Sections, replacing \mathcal{B}_W with its twisted version $\tilde{\mathcal{B}}_W$ where necessary.

5.11. Corollary. *Let $\mu: S(\mathfrak{h}) \rightarrow \mathcal{B}_W$ be as above, and let $W' = W(\text{supp } \mu(\mathfrak{h})) \subseteq W$. Then $\mu(f) = 0$ for any homogeneous W' -invariant polynomial $f \in S(\mathfrak{h})$ of positive degree.*

Proof. Take a homogeneous $f \in S(\mathfrak{h})_+^{W'}$. If a root α is in $\text{supp } \mu(\mathfrak{h})$ so that s_α is in W' , then $s_\alpha(f) = f$ and $f \overleftarrow{\partial}_\alpha = 0$, therefore $\mu(f) \overleftarrow{D}_{[\alpha]} = 0$ by Lemma 5.9. If $\alpha \notin \text{supp } \mu(\mathfrak{h})$, then $c_\alpha = 0$ by Lemma 5.5(2), hence $\mu(f) \overleftarrow{D}_{[\alpha]} = 0$ again by Lemma 5.9. Thus, $\mu(f)$ lies in the kernel of all $\overleftarrow{D}_{[\alpha]}$, which implies that $\mu(f) \in \mathbb{C}$ by Criterion 4.7. But since $\mu(f)$ is of positive degree, this means that $\mu(f) = 0$. \square

5.12. Remark. It follows from the Corollary that the kernel of μ contains the ideal $I_{W'}(\mathfrak{h}) := S(\mathfrak{h})S(\mathfrak{h})_+^{W'}$ of $S(\mathfrak{h})$. Let $U = \mu(\mathfrak{h})$; then μ induces a surjective map from $S(\mathfrak{h})/I_{W'}(\mathfrak{h})$ onto the subalgebra $\langle U \rangle$ of \mathcal{B}_W .

Note that \mathfrak{h} may not be the reflection representation for $W' = W(\text{supp } U)$ because $\mathfrak{h}' = \text{span}(\text{supp } U)$ is not necessarily the whole of \mathfrak{h} (although it is, if W is an irreducible Coxeter group). Still, $S(\mathfrak{h})/I_{W'}(\mathfrak{h})$ is isomorphic to the coinvariant algebra $S_{W'} = S(\mathfrak{h}')/I_{W'}$. Indeed, $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{k}$ where the action of W' on \mathfrak{k} is trivial; $S(\mathfrak{h}) = S(\mathfrak{h}') \oplus J$ and $I_{W'}(\mathfrak{h}) = I_{W'} \oplus J$ where $J = S(\mathfrak{h}')S(\mathfrak{k})_+$, so that the isomorphism follows.

Thus, we have already proved that there is an onto map $S_{W'} \rightarrow \langle U \rangle$. To complete the steps needed for the proof of Theorem 5.4, we have to show that this map is an isomorphism. We are going to use lemma 5.9 one more time.

5.13. Lemma. *In the above notation, the kernel of $\mu: S(\mathfrak{h}) \rightarrow \mathcal{B}_W$ is precisely $I_{W'}(\mathfrak{h}) = S(\mathfrak{h})S(\mathfrak{h})_+^{W'}$.*

Proof. The inclusion $I_{W'}(\mathfrak{h}) \subseteq \ker \mu$ has been demonstrated in the last Corollary and Remark. We assume now that $f \in S(\mathfrak{h})$ does not lie in $I_{W'}(\mathfrak{h})$ and show that $\mu(f) \neq 0$. Decompose $S(\mathfrak{h}) = S(\mathfrak{h}') \oplus J$ as in the Remark; since $J \subseteq I_{W'}(\mathfrak{h}) \subseteq \ker \mu$, it is enough to assume that $f \in S(\mathfrak{h}')$ (and $f \notin I_{W'}$). By 1.6, there exist roots $\gamma_1, \gamma_2, \dots, \gamma_l$ in the root system $\text{supp } U$, such that $f \overleftarrow{\partial}_{\gamma_1} \overleftarrow{\partial}_{\gamma_2} \dots \overleftarrow{\partial}_{\gamma_l} \in a + I_{W'}$ where $a \in S^0(\mathfrak{h})$ is a non-zero constant. Then by Lemma 5.9 one has $\mu(f) \overleftarrow{D}_{[\gamma_1]} \overleftarrow{D}_{[\gamma_2]} \dots \overleftarrow{D}_{[\gamma_l]} = c_{\gamma_1} c_{\gamma_2} \dots c_{\gamma_l} a$, which is not zero since $c_{\gamma_i} \neq 0$ by Lemma 5.5(2). Hence $\mu(f) \neq 0$. \square

5.14. Proof of the Theorem. The proof of Theorem 5.4 is already contained in 5.5–5.13, but we summarise it here for clarity. (i) If $U \subset V_W$ is a reflection submodule, write $U = \mu(\mathfrak{h})$ where $\mu: \mathfrak{h} \rightarrow V_W$ is a W -module map, and extend this map by 5.8 to a surjective homomorphism $\mu: S(\mathfrak{h}) \rightarrow \langle U \rangle$ of algebras. By Lemma 5.13, the kernel of μ in $S(\mathfrak{h})$ is $I_{W'}(\mathfrak{h})$ where $W' = W(\text{supp } U)$. Therefore, $\langle U \rangle$ is isomorphic to $S(\mathfrak{h})/I_{W'}(\mathfrak{h})$, which is $S_{W'}$ as shown in 5.12. (ii) Generic reflection submodules exist by 5.6. If U is such a submodule, i.e. $\text{supp } U = R$ and $W(\text{supp } U) = W$, then $\langle U \rangle \cong S_W$ by part (i). Theorem 5.4 is proved.

5.15. Automorphisms of the Nichols-Woronowicz algebra which permute copies of S_W in \mathcal{B}_W . We have seen that, if there is more than one W -orbit in the root system R of W , a degree-preserving embedding of S_W into \mathcal{B}_W is not unique. Such an embedding is determined by assigning a non-zero value of c_α to each W -orbit in R . However, two such

embeddings always differ by an action of an automorphism of \mathcal{B}_W . This easily follows from the explicit description of generic reflection submodules in V_W given in Lemma 5.5:

5.16. Lemma. *Let $\mu, \mu': S_W \hookrightarrow \mathcal{B}_W$ be two degree-preserving embeddings. There is a Hopf algebra automorphism $\theta: \mathcal{B}_W \rightarrow \mathcal{B}_W$ such that $\mu' = \theta \circ \mu$.*

Proof. We may assume μ to be fixed; let the restriction of μ to $\mathfrak{h} = S_W^1$ be given by $\mu(x) = \sum_{\alpha \in R} (x, \alpha)[\alpha]$. For an arbitrary embedding μ' one has $\mu'(x) = \sum_{\alpha \in R} c_\alpha (x, \alpha)[\alpha]$ with some non-zero coefficients c_α , W -invariant in α . Define an invertible linear map $\theta: V_W \rightarrow V_W$ by $\theta([\alpha]) = c_\alpha[\alpha]$. Then θ is an automorphism of V_W as a Yetter-Drinfeld module over W , because θ is compatible with both the W -action and the W -grading on V_W . Thus, θ preserves the braiding on V_W and therefore extends to a Hopf algebra automorphism $\theta: \mathcal{B}_W \rightarrow \mathcal{B}_W$. One has $\mu'(x) = \theta(\mu(x))$ for $x \in S_W^1$ and (since both sides are algebra homomorphisms) for all $x \in S_W$. \square

6. THE NILCOXETER SUBALGEBRA OF \mathcal{B}_W

In the previous section, we realised the coinvariant algebra S_W of the Coxeter group W as a subalgebra in the Nichols-Woronowicz algebra \mathcal{B}_W .

We now recall the nilCoxeter algebra N_W , which acts on S_W and is non-degenerately paired with S_W , as described in 1.3–1.5. We will now show that all this structure (the nilCoxeter algebra, its action on S_W and its pairing with S_W), and not only the algebra S_W itself, is realised in \mathcal{B}_W .

6.1. The subalgebra of \mathcal{B}_W isomorphic to the coinvariant algebra S_W , constructed in Section 5, depends on a choice of a W -invariant scalar function $\alpha \mapsto c_\alpha \neq 0$ on the root system R . From now on, we assume that

$$c_\alpha = 1 \quad \text{for all roots } \alpha,$$

to simplify the exposition. All results in this section may, however, be restated for arbitrary c_α by applying the automorphism $\theta: \mathcal{B}_W \rightarrow \mathcal{B}_W$, defined in the proof of Lemma 5.16.

6.2. In light of the above assumption, we consider a linear map

$$\mu: \mathfrak{h} \rightarrow V_W, \quad \mu(x) = \sum_{\alpha \in R} (x, \alpha)[\alpha],$$

that extends, as we know from Theorem 5.4 and its proof, to an embedding

$$\mu: S_W \hookrightarrow \mathcal{B}_W$$

of algebras.

By Lemma 5.9, the partial derivative operator $\overleftarrow{D}_{[\alpha]}$ corresponding to a root α acts on the subalgebra $\mu(S_W)$ as the divided difference operator $\overleftarrow{\partial}_\alpha$. Therefore, the restrictions $\overleftarrow{D}_{[\alpha_i]}|_{\mu(S_W)} \in \text{End } \mu(S_W)$ of the braided partial derivatives corresponding to simple roots $\alpha_1, \dots, \alpha_r$, onto the subalgebra $\mu(S_W)$ satisfy the nilCoxeter relations 1.3 just as the divided difference operators $\overleftarrow{\partial}_i$ do. It turns out that $\overleftarrow{D}_{[\alpha_i]}$ themselves (and not only their restrictions to the finite-dimensional subalgebra $\mu(S_W)$ of \mathcal{B}_W) satisfy the nilCoxeter relations. This is shown in the next Theorem.

Recall from 3.5 that the operators \overleftarrow{D}_x on the self-dual Nichols-Woronowicz algebra \mathcal{B}_W are defined for arbitrary $x \in \mathcal{B}_W$. If $x = [\alpha][\beta] \dots [\gamma]$ in \mathcal{B}_W , $\alpha, \beta, \dots, \gamma \in R$, one has $f \overleftarrow{D}_x = f \overleftarrow{D}_{[\alpha]} \overleftarrow{D}_{[\beta]} \dots \overleftarrow{D}_{[\gamma]}$ for $f \in \mathcal{B}_W$.

6.3. Theorem. (i) The simple root generators $[\alpha_1], \dots, [\alpha_r]$ in \mathcal{B}_W obey the nilCoxeter relations.

(ii) The map $\nu: N_W \rightarrow \mathcal{B}_W$, given on generators by $\nu(u_i) = [\alpha_i]$ and extended multiplicatively to N_W , is an algebra isomorphism between the nilCoxeter algebra N_W and the subalgebra of \mathcal{B}_W generated by $[\alpha_1], \dots, [\alpha_r]$.

(iii) The right action of N_W on S_W is expressed in terms of right derivations of \mathcal{B}_W : $\mu(fu) = \mu(f) \overleftarrow{D}_{\nu(u)}$ for $f \in S_W$, $u \in N_W$.

(iv) The non-degenerate pairing $\langle \cdot, \cdot \rangle_{S_W, N_W}$ coincides with the restriction of the self-duality pairing $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{B}_W}$ on \mathcal{B}_W to the subalgebras $\mu(S_W)$ and $\nu(N_W)$:

$$\langle f, u \rangle_{S_W, N_W} = \langle \mu(f), \nu(u) \rangle_{\mathcal{B}_W}.$$

6.4. Remark. As an illustration to this result, one may consider two diagrams:

$$\begin{array}{ccccc} S_W & \otimes & N_W & \rightarrow & S_W \\ \cap & & \cap & & \cap \\ \mathcal{B}_W & \otimes & \mathcal{B}_W & \rightarrow & \mathcal{B}_W \end{array} \quad \text{and} \quad \begin{array}{ccccc} S_W & \otimes & N_W & \rightarrow & \mathbb{C} \\ \cap & & \cap & & \parallel \\ \mathcal{B}_W & \otimes & \mathcal{B}_W & \rightarrow & \mathbb{C} \end{array},$$

where the horizontal arrows denote right action resp. pairing, and the inclusions \cap stand for the embeddings μ, ν of S_W and N_W into \mathcal{B}_W . The statement of the Theorem means that both diagrams are commutative.

6.5. We start the proof of Theorem 6.3 by verifying the Coxeter relation between the simple root generators $[\alpha_s]$ and $[\alpha_t]$ of \mathcal{B}_W . In fact, this is the longest part of the proof; for all crystallographic root systems, this can be achieved by an explicit calculation since it is enough to check the cases $m_{st} = 2, 3, 4, 6$. Our argument, however, is valid for any Coxeter group. Because of Woronowicz relations in \mathcal{B}_W , the Coxeter relation $[\alpha_s][\alpha_t][\alpha_s] \dots = [\alpha_t][\alpha_s][\alpha_t] \dots$ (m_{st} factors on each side) is equivalent to

$$[m_{st}]!_{\Psi}([\alpha_s] \otimes [\alpha_t] \otimes [\alpha_s] \otimes \dots) = [m_{st}]!_{\Psi}([\alpha_t] \otimes [\alpha_s] \otimes [\alpha_t] \otimes \dots),$$

where $[m_{st}]!_{\Psi}$ is the braided symmetriser defined in 2.6.

To prove this relation, we will express both sides explicitly in terms of paths in the Bruhat graph of a dihedral group.

6.6. The dihedral group. To check relation 6.5, we first note that we may restrict ourselves to the root subsystem of rank 2, generated by the simple roots α_s and α_t . This root subsystem will be of type $I_2(m)$ for some $m \geq 2$, and will consist of positive roots $\gamma_0 = \alpha_s, \gamma_1, \dots, \gamma_{m-2}, \gamma_{m-1} = \alpha_t$ and negative roots $\gamma_{m+i} = -\gamma_i, 0 \leq i \leq m-1$. These roots may be viewed as vectors $\gamma_i = (\cos \frac{i}{m}\pi, \sin \frac{i}{m}\pi)$ in the coordinate plane. Applying the reflection s_{γ_i} to a root

γ_j , one obtains γ_{m+2i-j} (the indices are understood modulo $2m$). The Coxeter group of type $I_2(m)$ is the dihedral group \mathbb{D}_m , which consists of the following elements:

$$\begin{aligned} v_0 &= \text{id}; & v_l &= s_{\gamma_0} s_{\gamma_{m-1}} s_{\gamma_0} \dots \text{ (} l \text{ factors), } & 1 \leq l \leq m; \\ v_{-l} &= s_{\gamma_{m-1}} s_{\gamma_0} s_{\gamma_{m-1}} \dots \text{ (} l \text{ factors), } & 1 \leq l \leq m; & & v_m = v_{-m}. \end{aligned}$$

The Coxeter generators of \mathbb{D}_m are $v_1 = s_{\gamma_0}$ and $v_{-1} = s_{\gamma_{m-1}}$. For any $l = 0, 1, \dots, m$, the length of $v_{\pm l}$ in the group \mathbb{D}_m is l . The group \mathbb{D}_m may be viewed as the parabolic subgroup of the Coxeter group W generated by the simple reflections with respect to α_s and α_t ; the length function $\ell(\cdot)$ on \mathbb{D}_m is then induced from W .

6.7. The Bruhat graph of the dihedral group. Recall that the Bruhat graph of a Coxeter group W is a labelled directed graph, with W as the set of vertices; an edge from w to w' exists if and only if $w' = s_\gamma w$ for a positive root γ of W , and $\ell(w') = \ell(w) + 1$; this edge is labelled by the root γ and denoted by $w' \xleftarrow{\gamma} w$.

We need an explicit description of the Bruhat graph of \mathbb{D}_m . This graph may be drawn as a regular $2m$ -gon with vertices $v_0, v_1, \dots, v_m, v_{-(m-1)}, \dots, v_{-1}$ (in this cyclic order) and sides parallel to roots γ_i . The edges of the graph are $v_{l+1} \xleftarrow{\gamma_l} v_l, v_{-(l+1)} \xleftarrow{\gamma_{m-l}} v_{-l}$ for $0 \leq l \leq m-1$ (the sides of the $2m$ -gon) and $v_{l+1} \xleftarrow{\gamma_0} v_{-l}, v_{-(l+1)} \xleftarrow{\gamma_{m-1}} v_l$ for $1 \leq l \leq m-2$ (the diagonals of the $2m$ -gon parallel to γ_0 or γ_{m-1}). In the Bruhat graph drawn this way, each edge is labelled by the positive root parallel to this edge.

We will consider paths in the Bruhat graph starting at the vertex $v_0 = \text{id}$. A path ω of length l , consisting of edges $v_{\pm l} \xleftarrow{\gamma_{i_l}} v_{\pm(l-1)} \dots \xleftarrow{\gamma_{i_2}} v_1 \xleftarrow{\gamma_{i_1}} v_0$, will be denoted by $\omega = (\gamma_{i_l}, \dots, \gamma_{i_2}, \gamma_{i_1})$.

6.8. The tensor representation of a Bruhat path. Consider an injective set-theoretical map

$$\begin{aligned} \{\text{paths in the Bruhat graph}\} & \xrightarrow{t} T(V_W) \\ \omega = (\gamma_{i_l}, \dots, \gamma_{i_2}, \gamma_{i_1}) & \mapsto t(\omega) = [\gamma_{i_l}] \otimes \dots \otimes [\gamma_{i_2}] \otimes [\gamma_{i_1}]. \end{aligned}$$

It is convenient to refer to $t(\omega)$ as the tensor representation of the path ω .

Let ω be a Bruhat path from the vertex $v_0 = \text{id}$ to a vertex $v \in \mathbb{D}_m$. The W -degree of $t(\omega) = [\gamma_{i_{\ell(v)}}] \otimes \dots \otimes [\gamma_{i_1}]$, i.e., the product $s_{\gamma_{i_{\ell(v)}}} \dots s_{\gamma_{i_1}}$, is equal to v , the final vertex of ω . Note that the braiding Ψ is compatible with the W -grading on $T(V_W)$: $\Psi_{i, i+1}$ leaves the W -degree intact. Thus, if we apply the braiding Ψ at positions $i, i+1$ to $t(\omega)$, we will get an element of $T(V_W)$ which is either a tensor representation of another path from v_0 to v or not a tensor representation of any Bruhat path at all. For example, $[\gamma_1] \otimes [\gamma_0]$ is the tensor representation of a Bruhat path from $v_0 = \text{id}$ to $v_2 = s_{\gamma_0} s_{\gamma_{m-1}}$, but one has $\Psi([\gamma_1] \otimes [\gamma_0]) = -[\gamma_2] \otimes [\gamma_1]$, which obviously does not correspond to any Bruhat path (when $m \geq 3$) because of the minus sign.

6.9. Ψ -generating paths. The path ω from $v_0 = \text{id}$ to $v \in \mathbb{D}_n$ will be called a Ψ -generating path, if the Woronowicz symmetrisation of the tensor representation of ω is the sum of tensor representations of all Bruhat paths from v_0 to v :

$$[\ell(v)]!_{\Psi} t(\omega) = \sum_{\omega' = v \leftarrow \dots \leftarrow v_0} t(\omega')$$

Note that there are $2^{\ell(v)-1}$ Bruhat paths from v_0 to v , since there are two choices for each intermediate vertex $v_{\pm 1}, \dots, v_{\pm \ell(v)-1}$ of the path. Thus, there are $2^{\ell(v)-1}$ terms on the right hand side of this equation. A priori, the left hand side has $\ell(v)!$ terms; hence, equality for $\ell(v) > 2$ may be possible only due to cancellations on the left.

Although the definition of a Ψ -generating Bruhat path makes sense for any Coxeter group W , in general we can only conjecture that Ψ -generating paths exist for any vertex v of the Bruhat graph of W . However, the case of the dihedral group \mathbb{D}_m is handled more easily because of a very explicit description of the Bruhat graph. We have the following

6.10. Lemma. (a) *The path $\omega_l^+ = (\gamma_0, \gamma_{m-1}, \gamma_0, \gamma_{m-1}, \dots)$ of length l is a Ψ -generating path from the vertex $v_0 = \text{id}$ to the vertex v_l in the Bruhat graph of the dihedral group \mathbb{D}_m .*
 (b) *The path $\omega_l^- = (\gamma_{m-1}, \gamma_0, \gamma_{m-1}, \gamma_0, \dots)$ of length l is a Ψ -generating path from v_0 to v_{-l} .*

Proof. Denote by P_l the sum of tensor representations of all Bruhat paths from v_0 to v_l . Throughout this proof, we are going to write γ_i instead of $[\gamma_i]$ for the basis elements of V_W ; this does not lead to a confusion but ensures that the generators of $T(V_W)$ do not mix up with the braided integers.

We have to show that $[l]!_{\Psi} t(\omega_l^{\pm}) = P_{\pm l}$. Induction in l ; the case $l = 0$ is trivial. If $l = 1$, $\omega_1^+ = (\gamma_0)$ is the only Bruhat path from $v_0 = \text{id}$ to $v_1 = s_{\gamma_0}$, and, trivially, it is Ψ -generating. Similarly for $\omega_1^- = (\gamma_{m-1})$.

Assume that $l \geq 2$ and that the Lemma is proved for $l-1$ and $l-2$. The properties $[l]!_{\Psi} = [l]_{\Psi}(\text{id} \otimes [l-1]!_{\Psi})$ and $[l]_{\Psi} = \text{id} + (\text{id} \otimes [l-1]_{\Psi})\Psi_{12}$ of braided integers and braided factorials follow from their definition 2.6. One therefore has

$$\begin{aligned} [l]!_{\Psi} t(\omega_l^+) &= [l]!_{\Psi} (\gamma_0 \otimes t(\omega_{l-1}^-)) = [l]_{\Psi} (\gamma_0 \otimes [l-1]!_{\Psi} t(\omega_{l-1}^-)) \\ &= [l]_{\Psi} (\gamma_0 \otimes P_{-(l-1)}) \\ &= \gamma_0 \otimes P_{-(l-1)} + (\text{id} \otimes [l-1]_{\Psi})\Psi_{12}(\gamma_0 \otimes P_{-(l-1)}). \end{aligned}$$

Any Bruhat path ω from v_0 to $v_{-(l-1)}$ passes either through the vertex $v_{-(l-2)}$ or through v_{l-2} , and the last edge of ω is labelled by γ_{m-l+1} or by γ_{m-1} , respectively. Therefore, $P_{-(l-1)} = \gamma_{m-l+1} \otimes P_{-(l-2)} + \gamma_{m-1} \otimes P_{l-2}$. Using this, we rewrite

$$\begin{aligned} [l]!_{\Psi} t(\omega_l^+) &= \gamma_0 \otimes P_{-(l-1)} \\ &\quad + (\text{id} \otimes [l-1]_{\Psi})(\Psi(\gamma_0 \otimes \gamma_{m-l+1}) \otimes P_{-(l-2)} + \Psi(\gamma_0 \otimes \gamma_{m-1}) \otimes P_{l-2}). \end{aligned}$$

We compute $\Psi(\gamma_0 \otimes \gamma_{m-l+1}) = \gamma_{l-1} \otimes \gamma_0$ and $\Psi(\gamma_0 \otimes \gamma_{m-1}) = \gamma_1 \otimes \gamma_0$. The tensors $P_{\pm(l-2)}$ are replaced, by the induction hypothesis, with $[l-2]!_{\Psi} t(\omega_{l-2}^{\pm})$. Thus we obtain

$$\begin{aligned} [l]!_{\Psi} t(\omega_l^+) &= \gamma_0 \otimes P_{-(l-1)} + \gamma_{l-1} \otimes [l-1]_{\Psi} (\gamma_0 \otimes [l-2]!_{\Psi} t(\omega_{l-2}^-)) \\ &\quad + \gamma_1 \otimes [l-1]_{\Psi} (\gamma_0 \otimes [l-2]!_{\Psi} t(\omega_{l-2}^+)) \\ &= \gamma_0 \otimes P_{-(l-1)} + \gamma_{l-1} \otimes [l-1]!_{\Psi} (\gamma_0 \otimes t(\omega_{l-2}^-)) + \gamma_1 \otimes [l-1]!_{\Psi} (\gamma_0 \otimes t(\omega_{l-2}^+)). \end{aligned}$$

Note that $\gamma_0 \otimes t(\omega_{l-2}^-) = t(\omega_{l-1}^+)$, so that by the induction hypothesis, the second term is equal to $\gamma_{l-1} \otimes P_{l-1}$. The tensor $\gamma_0 \otimes t(\omega_{l-2}^+)$ in the third term is of the form $\gamma_0 \otimes \gamma_0 \otimes \dots$, and lies in the kernel of the Woronowicz symmetriser $[l-1]!_{\Psi}$ (indeed, $[\gamma_0] \cdot [\gamma_0] \cdot \dots$ is zero in the Nichols-Woronowicz algebra \mathcal{B}_W). Therefore, the third term on the right hand side

vanishes, yielding $[l]!_{\Psi} t(\omega_l^+) = \gamma_0 \otimes P_{-(l-1)} + \gamma_{l-1} \otimes P_{l-1}$. But this is equal to P_l , because a path from v_0 to v_l in the Bruhat graph of \mathbb{D}_m either passes via $v_{-(l-1)}$ and has the last edge labelled by γ_0 , or passes via v_{l-1} and has the last edge labelled by γ_{l-1} .

An argument establishing the other equality $[l]!_{\Psi} t(\omega_l^-) = P_{-l}$, is completely analogous. The Lemma is thus proved. \square

6.11. Proof of the Coxeter relations. We are now ready to prove the Coxeter relation for the simple root generators $[\alpha_s] = [\gamma_0]$ and $[\alpha_t] = [\gamma_{m-1}]$ in the Nichols-Woronowicz algebra \mathcal{B}_W . Let us show that relation 6.5, which we rewrite as

$$[m]!_{\Psi}([\gamma_0] \otimes [\gamma_{m-1}] \otimes [\gamma_0] \otimes \dots) = [m]!_{\Psi}([\gamma_{m-1}] \otimes [\gamma_0] \otimes [\gamma_{m-1}] \otimes \dots),$$

holds. Indeed, since $v_m = v_{-m}$ (both are equal to the longest word in the group \mathbb{D}_m), Lemma 6.10 implies that both sides are equal to the sum of tensor representations of all paths from v_0 to v_m in the Bruhat graph of \mathbb{D}_m . The Coxeter relation is proved.

6.12. The rest of the proof of Theorem 6.3. To establish part (i) of the Theorem, it now remains to add that $[\alpha_i][\alpha_i] = 0$ in \mathcal{B}_W because $(\text{id} + \Psi)([\alpha_i] \otimes [\alpha_i]) = 0$.

By (i), there is a well-defined algebra homomorphism $\nu: N_W \rightarrow \mathcal{B}_W$ defined by $\nu(u_i) = [\alpha_i]$. Let $u = u_{i_1} u_{i_2} \dots u_{i_l}$ be a basis element of N_W . Then $\overleftarrow{D}_{\nu(u)}$ is $\overleftarrow{D}_{[\alpha_{i_1}]} \dots \overleftarrow{D}_{[\alpha_{i_l}]}$, and by Lemma 5.9 $\mu(f) \overleftarrow{D}_{\nu(u)} = \mu(fu)$, so part (iii) of the theorem follows. Part (iv) also follows because $\langle f, u \rangle_{S_W, N_W}$ is the constant term of fu in S_W , which is equal to the constant term of $\mu(fu)$ in \mathcal{B}_W ; the latter is $\epsilon(f \overleftarrow{D}_{\nu(u)})$ which equals $\langle \mu(f), \nu(u) \rangle_{\mathcal{B}_W}$ by 3.5. We are left to prove part (ii); but (iv) implies that the image of ν is non-degenerately paired with the $|W|$ -dimensional subalgebra $\mu(S_W)$ in \mathcal{B}_W , therefore $\dim(\text{im } \nu) = |W|$ and ν is one-to-one. This finishes the proof of Theorem 6.3.

6.13. Remark. Although we proved Coxeter relations between generators of the Nichols-Woronowicz algebra \mathcal{B}_W corresponding to simple roots, the same method shows that any two generators $[\alpha]$ and $[\beta]$ of \mathcal{B}_W , $\alpha, \beta \in R$, obey a Coxeter relation up to a sign. Indeed, consider the root subsystem of type $I_2(m)$ generated by α and β , where $m \geq 2$ is such that the scalar product (α, β) equals $-c \cos \frac{\pi}{m}$, $c = \pm 1$. The roots $\alpha, c\beta$ may be chosen as the simple roots in this root subsystem; the above argument allows one to compute the Woronowicz symmetriser of $[\alpha] \otimes [c\beta] \otimes [\alpha] \otimes \dots$ and yields the Coxeter relation of degree m between $[\alpha]$ and $c[\beta]$.

In particular, if γ is the highest root of a crystallographic root system R , the generators $[\alpha_1], \dots, [\alpha_r], [-\gamma]$ obey the affine nilCoxeter relations and generate a subalgebra in \mathcal{B}_W which is a quotient of the (infinite-dimensional) ‘affine nilCoxeter algebra’. However, this quotient is proper and finite-dimensional in known cases. This observation is due to A. N. Kirillov.

7. THE ALGEBRAS \mathcal{B}_W AND THE CONSTRUCTIONS OF FOMIN-KIRILLOV AND KIRILLOV-MAENO

We conclude the paper by outlining the relationship between the Nichols algebra \mathcal{B}_W which we constructed for an arbitrary Coxeter group W , the quadratic algebra \mathcal{E}_n constructed in [FK] for the symmetric group \mathbb{S}_n , and the generalisation $BE(W, S)$ of \mathcal{E}_n for an arbitrary Coxeter group, defined in [KM1].

7.1. The quadratic algebra $\mathcal{B}_{quad}(V_W)$. Let $\Psi: V_W \otimes V_W \rightarrow V_W \otimes V_W$ be the braiding on the Yetter-Drinfeld module V_W defined in Section 4, and let $T(V_W)$ be the free braided group. Denote by $I_{quad}(V_W)$ the two-sided ideal of $T(V_W)$ generated by $\ker(\text{id} + \Psi) \subset V_W^{\otimes 2}$. Put

$$\mathcal{B}_{quad}(V_W) = T(V_W)/I_{quad}(V_W);$$

that is, to define $\mathcal{B}_{quad}(V_W)$, one imposes only the quadratic Woronowicz relations on $T(V_W)$. The algebra \mathcal{B}_{quad} is a braided Hopf algebra in the category ${}^W_W\mathcal{YD}$ with a self-duality pairing which may be degenerate; the Nichols algebra \mathcal{B}_W is a (possibly proper) quotient of $\mathcal{B}_{quad}(V_W)$.

7.2. $\mathcal{B}_{quad}(V_{\mathbb{S}_n})$ is the Fomin-Kirillov algebra. The algebra $\mathcal{B}_{quad}(V_{\mathbb{S}_n})$ is the same as the quadratic algebra \mathcal{E}_n , introduced by Fomin and Kirillov in [FK]. This was independently observed in [MiS] and in [M5]. These algebras coincide as braided Hopf algebras in the Yetter-Drinfeld module category over \mathbb{S}_n . For $1 \leq a < b \leq n$, let ab denote the root $\alpha_a + \alpha_{a+1} + \dots + \alpha_{b-1}$ in the root system of \mathbb{S}_n ; the operators $\Delta_{ab}: \mathcal{E}_n \rightarrow \mathcal{E}_n$, defined in [FK, Section 9], can be viewed as braided partial derivatives on $\mathcal{B}_{quad}(V_{\mathbb{S}_n})$, as was noticed in [M5]. It is also shown in [M5] that the Hopf algebra structure on the ‘twisted group algebra’ $\mathcal{E}_n \otimes \mathbb{C}\mathbb{S}_n$, introduced and studied in [FP], can be obtained by Majid’s biproduct bosonisation of \mathcal{E}_n .

7.3. \mathcal{B}_W and Kirillov-Maeno bracket algebras. For a Coxeter group W , the bracket algebra $BE(W, S)$, where S stands for the set of Coxeter generators, is defined in [KM1] as the quotient of the tensor algebra $T(V_W)$ of the linear space V_W by the following relations (we use our notation from Sections 1–4):

- (1) $[\gamma]^2 = 0$ for all $\gamma \in R$;
- (2) For any intersection R' of a 2-dimensional plane in \mathfrak{h} with R , let the roots in R' be $\gamma_0, \gamma_1, \dots, \gamma_{2m-1}$ enumerated as in 6.6. The relations

$$\begin{aligned} \sum_{i=0}^{m-1} [\gamma_i][\gamma_{i+k}] &= 0 \quad \text{for all } k; \\ [\gamma_l] \cdot [\gamma_0][\gamma_1] \dots [\gamma_{2l}] &+ [\gamma_0][\gamma_1] \dots [\gamma_{2l}] \cdot [\gamma_l] + [\gamma_l] \cdot [\gamma_{2l}][\gamma_{2l-1}] \dots [\gamma_0] \\ &+ [\gamma_{2l}][\gamma_{2l-1}] \dots [\gamma_0] \cdot [\gamma_l] = 0 \quad \text{for } l = [m/2] - 1, \end{aligned}$$

are imposed in $BE(W, S)$. The second, 4-term relation is meaningful only when $m \geq 4$.

The bracket algebras generalise the quadratic algebras \mathcal{E}_n to the case of arbitrary Coxeter group. If W is a Weyl group of simply-laced type, one has $BE(W, S) = \mathcal{B}_{quad}(V_W)$ because there are no 4-term relations in $BE(W, S)$. The relations in the bracket algebra are just sufficient to prove that $BE(W, S)$ contains a commutative subalgebra isomorphic to the coinvariant algebra S_W , which was done in [KM1] for crystallographic Coxeter groups of classical type and of type G_2 . However, in some cases the bracket algebra has ‘less’ relations than \mathcal{B}_W has.

For example, when W is the Weyl group of type B_2 , one has $\dim BE(W, S) = \infty$ according to [KM1]; $\dim \mathcal{B}_W = 64$ which may be verified by a computer calculation. In this case, \mathcal{B}_W is the quotient of $BE(W, S)$ by the Coxeter relation $[\alpha_1][\alpha_2][\alpha_1][\alpha_2] = [\alpha_2][\alpha_1][\alpha_2][\alpha_1]$ between the simple root generators.

7.4. Proposition. *If W is a Weyl group of type other than G_2 , the Nichols algebra \mathcal{B}_W is a quotient of the bracket algebra $BE(W, S)$.*

Proof. The relation (1) of the bracket algebra holds in \mathcal{B}_W . Now let the root subsystem $R' = \{\gamma_0, \dots, \gamma_{2m-1}\}$ be as in (2). One checks (see 6.6) that $\Psi([\gamma_i] \otimes [\gamma_j]) = -[\gamma_{2i-j}] \otimes [\gamma_i]$ where Ψ is the braiding 4.2 on V_W . Note that $\gamma_{m+i} = -\gamma_i$ as the indices are taken modulo $2m$. Applying $\text{id} + \Psi$ to the left hand side of the quadratic relation in (2), one gets $\sum_{i=0}^m [\gamma_{i+k}] \otimes [\gamma_i] - [\gamma_{i+2k}] \otimes [\gamma_{i+k}]$ which is zero. Thus, the quadratic relation in (2) holds in \mathcal{B}_W because its left hand side lies in $\ker(\text{id} + \Psi)$.

It remains to show that the 4-term relation in (2) holds in \mathcal{B}_W . If W is of type A , $B = C$, D , E or F , a root subsystem $R' \subset R$ of rank 2 consists of at most 8 roots, i.e. $m \leq 4$. For $m = 4$ and $l = 1$, the braided symmetriser $[4]!_{\Psi}$, applied to the left hand side of the 4-term relation, gives zero — this is verified by easy computation using factorisation 2.6 of $[4]!_{\Psi}$. \square

It has been observed by T. Maeno that in type G_2 , the 4-term relation in the bracket algebra is not compatible with the braided Hopf algebra structure and therefore cannot hold in the Nichols-Woronowicz algebra. Thus, the statement of the Proposition is not true when W is of type G_2 .

7.5. The intriguing question remains, whether the Nichols algebra \mathcal{B}_{S_n} coincides with the quadratic algebra \mathcal{E}_n or is a proper quotient of it.

The graded components of degrees 1, 2, 3 in \mathcal{B}_{S_n} and \mathcal{E}_n may be shown to coincide. Furthermore, $\mathcal{B}_{S_n} = \mathcal{E}_n$ for $n \leq 5$ (see [MiS, Example 6.4] for $n \leq 4$, [G] for $n = 5$). Incidentally, S_n for $n \leq 5$ and W_{B_2} are the only examples of Coxeter groups where we know the Nichols algebra \mathcal{B}_W to be finite-dimensional.

We finish with the following conjecture, which already appeared in a number of sources including [MiS] and [M5]. If true, this conjecture would mean that our construction of \mathcal{B}_W as a model for the Schubert calculus generalises, in proper sense, the Fomin-Kirillov construction.

7.6. Conjecture. *The algebras \mathcal{B}_{S_n} are quadratic and coincide with the Fomin-Kirillov algebras \mathcal{E}_n for all n .*

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